Во задачите 1-4 ќе ја користиме границата

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$$

1а) $\lim\limits_{x\to 0}\frac{\sin 4x}{x}=\lim\limits_{x\to 0}\frac{4\sin 4x}{4x}=4\cdot 1=4$

б) $\lim\limits_{x\to 0}\frac{\sin x}{5x}=\frac{1}{5}$

в) $\lim\limits_{x\to 0}\frac{\sin 8x}{3x}=\lim\limits_{x\to 0}\frac{8x\cdot\frac{\sin 8x}{8x}}{3x}=\lim\limits_{x\to 0}\frac{8x}{3x}=\frac{8}{3}$

г) $\lim\limits_{x\to 0}\frac{\sin\frac{x}{3}}{2x}=\lim\limits_{x\to 0}\frac{\frac{x}{3}\cdot\frac{\sin\frac{x}{3}}{\frac{x}{3}}}{2x}=\lim\limits_{x\to 0}\frac{\frac{x}{3}}{\frac{2x}{1}}=\lim\limits_{x\to 0}\frac{x}{6x}=\frac{1}{6}$

или: $\lim\limits_{x\to 0}\frac{\sin\frac{x}{3}}{2x}=\lim\limits_{x\to 0}\frac{\sin\frac{x}{3}}{2\cdot\frac{x}{3}\cdot 3}=\frac{1}{6}$

2. а) $\lim\limits_{x\to 0}\frac{\sin 4x-\sin 2x}{x}=\lim\limits_{x\to 0}\left(\frac{\sin 4x}{x}-\frac{\sin 2x}{x}\right)=$

$=\lim\limits_{x\to 0}\left(4\cdot\frac{\sin 4x}{4x}-2\cdot\frac{\sin 2x}{2x}\right)=4-2=2$

б) $\lim\limits_{x\to 0}\frac{\sin 9x}{\sin 5x}=\lim\limits_{x\to 0}\frac{9x\cdot\frac{\sin 9x}{9x}}{5x\cdot\frac{\sin 5x}{5x}}=\lim\limits_{x\to 0}\frac{9x}{5x}=\frac{9}{5}$

в) $\lim\limits_{x\to 0}\frac{\sin 4x-\sin 2x}{\sin x}=\lim\limits_{x\to 0}\left(\frac{\sin 4x}{\sin x}-\frac{\sin 2x}{\sin x}\right)=$

$=\lim\limits_{x\to 0}\left(\frac{4x\cdot\frac{\sin 4x}{4x}}{x\cdot\frac{\sin x}{x}}-\frac{2x\cdot\frac{\sin 2x}{2x}}{x\cdot\frac{\sin x}{x}}\right)=\lim\limits_{x\to 0}\left(\frac{4x}{x}-\frac{2x}{x}\right)=4-2=2$

г) $\lim\limits_{x\to 0} x\operatorname{ctg}5x = \lim\limits_{x\to 0} x\dfrac{\cos 5x}{\sin 5x} = \lim\limits_{x\to 0} x\dfrac{\cos 5x}{\dfrac{5x\sin 5x}{5x}} =$ ($\dfrac{\sin 5x}{5x}\to 1$)

$= \lim\limits_{x\to 0} \cancel{x}\dfrac{\cos 5x}{5\cancel{x}} = \lim\limits_{x\to 0}\dfrac{\cos 5x}{5} = \dfrac{\cos 0}{5} = \dfrac{1}{5}$

3а) $\lim\limits_{x\to 0}\dfrac{\operatorname{tg}4x}{\sin x} = \lim\limits_{x\to 0}\dfrac{\frac{\sin 4x}{\cos 4x}}{\frac{\sin x}{1}} = \lim\limits_{x\to 0}\dfrac{\sin 4x\cdot 1}{\cos 4x\cdot\sin x} =$

$= \lim\limits_{x\to 0}\dfrac{4x\cdot\frac{\sin 4x}{4x}}{\cos 4x\cdot x\cdot\frac{\sin x}{x}} = \lim\limits_{x\to 0}\dfrac{4x}{x\cdot\cos 4x} = \lim\limits_{x\to 0}\dfrac{4}{\cos 4x} = \dfrac{4}{\cos 0} = 4$

($\frac{\sin 4x}{4x}\to 1$, $\frac{\sin x}{x}\to 1$, $\cos 0 = 1$)

б) $\lim\limits_{x\to 0}\dfrac{\sin^2 2x}{4x^2} = \lim\limits_{x\to 0}\left(\dfrac{\sin 2x}{2x}\right)^2 = 1^2 = 1$

в) $\lim\limits_{x\to 0}\dfrac{\sin^2 2x}{x^2} = \lim\limits_{x\to 0}\dfrac{2^2\sin^2 2x}{2^2x^2} = \lim\limits_{x\to 0} 4\dfrac{\sin^2 2x}{(2x)^2} = 4$

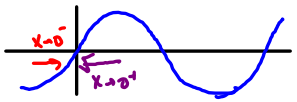
($\frac{\sin^2 2x}{(2x)^2}\to 1$)

или: $\lim\limits_{x\to 0}\dfrac{\sin^2 2x}{x^2} = \lim\limits_{x\to 0}\dfrac{\sin 2x\cdot\sin 2x}{x\cdot x} = \lim\limits_{x\to 0} 2\dfrac{\sin 2x}{2x}\cdot 2\dfrac{\sin 2x}{2x} = 4$

($\frac{\sin 2x}{2x}\to 1$, $\frac{\sin 2x}{2x}\to 1$)

ґ) $\lim\limits_{x\to 0}\dfrac{x^2}{\sin^3 2x} = \lim\limits_{x\to 0}\dfrac{(2x)^2}{4\sin^2 2x\cdot\sin 2x} = \lim\limits_{x\to 0}\dfrac{1}{4\sin 2x} =$ ($\frac{(2x)^2}{\sin^2 2x}\to 1$)

$= \dfrac{1}{4\sin 2\cdot 0} = \dfrac{1}{4\sin 0} = ?$ ($\sin 0 \to 0$)



$\lim\limits_{x\to 0^+}\dfrac{x^2}{\sin^3 2x} = \dfrac{1}{4\sin 0} = +\infty$ ($\sin 0 \to 0^+$)

$\lim\limits_{x\to 0^-}\dfrac{x^2}{\sin^3 2x} = \dfrac{1}{4\sin 0} = -\infty$ ($\sin 0 \to 0^-$)

4 а) $\lim\limits_{x\to\infty} x\sin\frac{1}{x} = (*)$

воведување смена

$\frac{1}{x} = t$ Ако $x \to \infty$, тогаш

$x = \frac{1}{t}$ $t = \frac{1}{x} \to 0$

$(*) = \lim\limits_{t\to 0} \frac{1}{t}\sin t = 1$ ($\frac{\sin t}{t} \to 1$)

б) $\lim\limits_{x\to 0} \frac{x}{\sqrt{\sin x}} = \lim\limits_{x\to 0} \frac{x}{\sqrt{\sin x}} \cdot \frac{\sqrt{\sin x}}{\sqrt{\sin x}} = \lim\limits_{x\to 0} \frac{x\sqrt{\sin x}}{(\sqrt{\sin x})^2} =$

$= \lim\limits_{x\to 0} \frac{x\sqrt{\sin x}}{\sin x} = \lim\limits_{x\to 0} \frac{\sqrt{\sin x}}{\frac{\sin x}{x}} = \lim\limits_{x\to 0} \sqrt{\sin x} = \sqrt{\sin 0} = \sqrt{0} = 0$ ($\frac{\sin x}{x} \to 1$)

Во задачите 5 и 6 ќе ги користиме границите

$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$ $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$

5 а) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x+1} = \lim\limits_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^x\right)^2\left(1+\frac{1}{x}\right)^1 =$ ($\left(1+\frac{1}{x}\right)^x \to e$, $\frac{1}{x} \to 0$)

$= e^2 \cdot 1 = e^2$

б) $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}\cdot 2} =$

$= \lim\limits_{x\to\infty}\left(\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}}\right)^2 = e^2$ ($\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}} \to e$)

(Задачата може да се реши и со смена како што е објаснето во задача 5 б)

б) $\lim\limits_{x\to\infty}\left(1+\frac{5}{x}\right)^{x+1}=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{5}}\right)^{x+1}=(*)$

Введуємо заміну

$\frac{x}{5}=t$

$x=5t$

Якщо $x\to\infty$, тоді $t=\frac{x}{5}\to\infty$

$(*)=\lim\limits_{t\to\infty}\left(1+\frac{1}{t}\right)^{5t+1}=\lim\limits_{t\to\infty}\left(\left(1+\frac{1}{t}\right)^{t}\right)^{5}\left(1+\frac{1}{t}\right)^{1}=e^5$ ($\left(1+\frac{1}{t}\right)^{t}\to e$, $\frac{1}{t}\to 0$)

в) $\lim\limits_{x\to\infty}\left(\frac{x}{x-3}\right)^{x}=\lim\limits_{x\to\infty}\left(\frac{x-3+3}{x-3}\right)^{x}=\lim\limits_{x\to\infty}\left(\frac{x-3}{x-3}+\frac{3}{x-3}\right)^{x}=$

$=\lim\limits_{x\to\infty}\left(1+\frac{3}{x-3}\right)^{x}=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x-3}{3}}\right)^{x}=(*)$

заміна $\frac{x-3}{3}=t$

$x-3=3t$

$x=3t+3$

Якщо $x\to\infty$, тоді $t=\frac{x-3}{3}=\frac{\infty-3}{3}=\frac{\infty}{3}\to\infty$

$(*)=\lim\limits_{t\to\infty}\left(1+\frac{1}{t}\right)^{3t+3}=\lim\limits_{t\to\infty}\left(\left(1+\frac{1}{t}\right)^{t}\right)^{3}\left(1+\frac{1}{t}\right)^{3}=$ ($\left(1+\frac{1}{t}\right)^{t}\to e$, $\frac{1}{t}\to 0$)

$=e^3\cdot 1=e^3$

а) $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(\frac{x+1-1}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(\frac{x+1}{x+1}-\frac{1}{x+1}\right)^x =$

$= \lim\limits_{x\to\infty}\left(1+\frac{1}{-(x+1)}\right)^x = (*)$

Замена

$-(x+1)=t$

$-x-1=t$

$x=-t-1$

Так $x\to\infty$, тогда

$t=-(x+1)\to-\infty$

$(*) = \lim\limits_{t\to-\infty}\left(1+\frac{1}{t}\right)^{-t-1} = \lim\limits_{t\to-\infty}\left(\left(1+\frac{1}{t}\right)^t\right)^{-1}\left(1+\frac{1}{t}\right)^{-1} =$ ($\left(1+\frac{1}{t}\right)^t \to e$, $\frac{1}{t}\to 0$)

$= e^{-1}\cdot 1^{-1} = \frac{1}{e}\cdot\frac{1}{1} = \frac{1}{e}$

б) $\lim\limits_{x\to\infty}\left(\frac{x-2}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(\frac{x+1-1-2}{x+1}\right)^x =$

$= \lim\limits_{x\to\infty}\left(\frac{x+1}{x+1}+\frac{-3}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{-3}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x+1}{-3}}\right)^x$

Замена $\frac{x+1}{-3}=t$

$x+1=-3t$

$x=-3t-1$

Так $x\to\infty$, тогда

$t=\frac{x+1}{-3}=\frac{\infty}{-3}\to-\infty$

$(*) = \lim\limits_{t\to-\infty}\left(1+\frac{1}{t}\right)^{-3t-1} = \lim\limits_{t\to-\infty}\left(\left(1+\frac{1}{t}\right)^t\right)^{-3}\left(1+\frac{1}{t}\right)^{-1} =$ ($\left(1+\frac{1}{t}\right)^t \to e$, $\frac{1}{t}\to 0$)

$= e^{-3}\cdot 1^{-1} = \frac{1}{e^3}\cdot\frac{1}{1} = \frac{1}{e^3}$